AMERICAN UNIVERSITY OF SHARJAH

ELE 353: CONTROL SYSTEMS I

PROJECT REPORT: CTE DOCUMENT 2

The Fox H-Function in Control Systems

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The Fox H-Function in Control Systems

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Abstract

This is a continuous time evaluation (CTE) documment that highlights the progress in the course project for the module ELE 353 - Control Systems I, and is submitted in partial fulfilment of the requirement for Midterm II of the course. The document delineates precise objectives of the project along with the goal statement, and outlines the results that motivated the team to proceed in the direction which shaped the research. It provides the necessary mathematical background in integral transform theory and the Fox H-Function, and explains the interconnection between all the presented concepts, and their application to control systems. It also presents a derivation of the associated formulae, and provides analytical and numerical results of simulations and implementations to verify the accuracy of the obtained expressions.

I. Introduction and Goal Statement

This document is a continuation of CTE Document I submitted on October 21, 2018. It picks off from the progress then and presents the research and results done in the period between Midterm 1 and Midterm 2.

In CTE 1, the goal of the project was presented as applying hypergeometric functions to control systems. Although this statement is vague, it summarizes the gist of the project. This goal has been refined and updated based on extensive mathematical research and analysis, which motivated the utilization of the Fox H-Function in particular.

Further to these updates, the goal of the project is to *analyze the response of Linear and Time Invariant (LTI) systems for a variety of inputs, inclusive of the step response, the impulse response, and the sine response, and provide explicit, novel and exact formulae for each of these in terms of the Fox H-Function*. Doing so would be a step in developing a theory wherein the complexity of analysis of subsystems becomes independent of the order of the system.

II. Objective

The theoretical nature of the project presents a plethora of opportunities with respect to achieving the goal of this project. This section highlights a well defined objective for this CTE document. It has been written with the objective of conveying the realtionship between the theory of hypergeometric functions and control systems. It aims to do so by exploting the established relationshipw between the various integral transforms, and manipulating the transfer functions to be in a form that can be represented by the Fox H-Function. A step-by-step procedure on how this is done is provided in Section III, and verified in Section V. This is in accordance with the goal of achiving an H-Function representation for the response of an arbitrary LTI system of given order for various test inputs.

III. Results

A. Mathematical Concepts

This section introduces some of the basic notation, terminology, and defnitions used in this paper.

1) Fox H-Function: The Fox H-Function of a scalar complex variable z, henceforth referred to as the H-Function is defined in terms of a Mellin-Barnes integral as shown in Equation 1.

$$\begin{aligned} H_{p,q}^{m,n}\left(z \left| \begin{matrix} (a_1, A_1), \dots, (a_p, A_p) \\ (b_1, B_1), \dots, (b_q, B_q) \end{matrix} \right.\right) &= H_{p,q}^{m,n}\left(z \left| \begin{matrix} (a_i, A_i)_{1:p} \\ (b_j, B_j)_{1:q} \end{matrix} \right.\right) = H_{p,q}^{m,n}(z) = H(z) \\ &= \frac{1}{2\pi w} \int_L \frac{\prod_{j=1}^{m} \Gamma(b_j + B_j s) \prod_{i=1}^{n} \Gamma(1 - a_i - A_i s)}{\prod_{i=n+1}^{p} \Gamma(a_i + A_i s) \prod_{j=m+1}^{q} \Gamma(1 - b_j - B_j s)} z^{-s} ds \end{aligned} \tag{1}$$

Here, $\{m, n, p, q\} \subset \mathbb{N}$ with $0 \leq n \leq p$, $1 \leq m \leq q$, $A_i, B_j \in \mathbb{R}^+$, $a_i, b_j \in \mathbb{C}\ \forall i, j$; $w = \sqrt{-1}$, $z \neq 0$, and $z^{-s} = \exp\{-s(\ln|z| + i \arg z)\}$ where $|\cdot|$ is the absolute value operator and $\arg z$ is not necessarily the principal value. L is a contour on the complex s plane that runs from $c_1 - i\infty$ to $c_2 + i\infty$, $c_1, c_2 \in \mathbb{R}$ such that it separates the poles of the $\Gamma(b_j + B_j s)$ terms in the numerator from those of the $\Gamma(1 - a_i - A_i s)$ terms in the numerator. Hence, L must separate

$$\zeta_{jv} = -\left(\frac{b_j + v}{B_j}\right),\ j \in [1, m] \cap \mathbb{Z};\ v \in \mathbb{N}_0 \tag{2}$$

from

$$\omega_{\lambda k} = \left(\frac{1 - a_\lambda + k}{A_\lambda}\right),\ \lambda \in [1, m] \cap \mathbb{Z};\ k \in \mathbb{N}_0 \tag{3}$$

where $\cap$ is the set intersection operator and $\mathbb{N}_0$ is the set of all postive integers including zero.

2) Mellin Transform: The Mellin Transform is an integral transform defined as follows. Let $f(t)$ represent a real valued function on $(0, \infty)$, which is essentially the non-transformed space (time domain). The Mellin Transform of $f(t)$ is given as:

$$F(s) = \mathcal{M}\{f(t)\} = \int_0^\infty f(t) t^{s-1} dt \tag{4}$$

This transform is invertible, and the Inverse Mellin Transform of $F(s)$ is given as:

$$f(t) = \mathcal{M}^{-1}\{F(s)\} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} F(s) t^{-s} ds \tag{5}$$

3) Laplace Transform: The Laplace Transform is an integral transform defined as follows. Let $f(t)$ represent a real valued function on $(-\infty, \infty)$. The Bilateral Laplace Transform of $f(t)$ is given as:

$$F(s) = \mathcal{B}\{f(t)\} = \int_{-\infty}^\infty f(t) e^{-st} dt \tag{6}$$

However, Bilateral Transforms do not respect causality, and control systems employ the Unilateral Laplace Transform, defined as

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty f(t) e^{-st} dt \tag{7}$$

This transform is invertible, and the Inverse Laplace Transform of $F(s)$ is given as:

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} F(s) e^{st} ds \tag{8}$$

B. Relationships between Integral Transforms

This section highlights the important relationships between integral trasnforms that are utilized in this paper.

1) Relationship between Unilateral and Bilateral Laplace Transform: It is evident that $\mathcal{L}\{f(t)\} = \mathcal{B}\{f(t)u(t)\}$ where $u(t)$ is the Heaviside unit step function. This can be easily proven as follows:

$$\begin{aligned}\mathcal{B}\{f(t)u(t)\} &= \int_{-\infty}^{\infty} f(t)u(t)e^{-st}dt \\ &= \int_{-\infty}^{0^-} f(t)u(t)e^{-st}dt + \int_{0^+}^{\infty} f(t)u(t)e^{-st}dt \\ &= \int_{-\infty}^{0^-} f(t)(0)e^{-st}dt + \int_{0^+}^{\infty} f(t)(1)e^{-st}dt \\ &= \mathcal{L}\{f(t)\} \end{aligned} \tag{9}$$

This integral converges as countably many finite discontinuities do not affect the Riemann integral value.

2) Relationship between Bilateral Laplace Transform and Mellin Transform: The Bilateral Laplace Transform and the Mellin Transform are related as:

$$\mathcal{B}\{f(t)\} = \mathcal{M}\{f(-\log(t))\} \tag{10a}$$

$$\mathcal{M}\{f(t)\} = \mathcal{B}\{f(e^{-t})\} \tag{10b}$$

C. Impulse Response of Linear and Time Invariant Systems

This section deals with Linear and Time Invariant (LTI) systems, and the representation of transfer functions as a Fox H-Function. Let t represent the variable time in the non-transformed space. Let s be the transformed variable in Laplace domain. In this discussion, an LTI system is understood to have the following transfer function, with r zeros and l poles including multiplicity:

$$G(s) = \frac{C(s)}{R(s)} = \frac{(s+z_1)(s+z_2)...(s+z_r)}{(s+p_1)(s+p_2)...(s+p_l)} \tag{11}$$

Here, $C(s)$ and $R(s)$ represent the system output and input in the Laplace Domain, while $-z_i, -p_j \in \mathbb{C}$ are the zeros and poles of the transfer function respectively.

The impulse response of the system is then given as

$$g(t) = \mathcal{L}^{-1}\{G(s)\} \tag{12}$$

We now define another function, $f(\cdot)$ as follows:

$$f(x) := g(-\log(x)) \tag{13a}$$

$$g(x) := f(e^{-x}) \tag{13b}$$

This motivates the following:

$$\begin{aligned} G(s) &= \mathcal{L}\{g(t)\} && \text{(14a)}\\ &= \mathcal{B}\{g(t)u(t)\} && \text{(14b)}\\ &= \mathcal{B}\{f(e^{-t})u(e^{-t})\} && \text{(14c)}\\ &= \mathcal{B}\{f(e^{-t})\} && \text{(14d)}\\ &= \mathcal{M}\{f(t)\} && \text{(14e)}\\ & && \text{(14f)}\end{aligned}$$

Therefore, we infer

$$f(t) = \mathcal{M}^{-1}\{G(s)\} \tag{15a}$$
$$= \mathcal{M}^{-1}\left\{\frac{(s+z_1)(s+z_2)...(s+z_r)}{(s+p_1)(s+p_2)...(s+p_l)}\right\} \tag{15b}$$
$$= \mathcal{M}^{-1}\left\{\frac{\prod_{i=1}^{r}(s+z_i)}{\prod_{i=1}^{l}(s+p_i)}\right\} \tag{15c}$$
$$= \mathcal{M}^{-1}\left\{\frac{\prod_{i=1}^{r}\frac{\Gamma(s+z_i+1)}{\Gamma(s+z_i)}}{\prod_{j=1}^{l}\frac{\Gamma(s+p_j+1)}{\Gamma(s+p_j)}}\right\} \tag{15d}$$
$$= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\prod_{i=1}^{r}\Gamma(z_i+1+s)\prod_{j=1}^{l}\Gamma(p_j+s)}{\prod_{j=1}^{l}\Gamma(p_j+1+s)\prod_{i=1}^{r}\Gamma(z_i+s)}t^{-s}ds \tag{15e}$$
$$= H_{l+r,l+r}^{l+r,0}\left(t\left|\begin{array}{l}(1+p_j,1)_{1:l},(z_i,1)_{1:r}\\(1+z_i,1)_{1:r},(p_j,1)_{1:l}\end{array}\right.\right) \tag{15f}$$

Finally,

$$g(t) = f(e^{-t})$$
$$= H_{l+r,l+r}^{l+r,0}\left(e^{-t}\left|\begin{array}{l}(1+p_j,1)_{1:l},(z_i,1)_{1:r}\\(1+z_i,1)_{1:r},(p_j,1)_{1:l}\end{array}\right.\right) \tag{16}$$

This analytically and immediately provides an exact, closed-form expression for the impulse response of any LTI system, regardless of the order.

D. Output of LTI System for Test Functions

1) Unit Step Response: Let the input $r(t) = u(t)$. Therefore, we have $R(s) = \frac{1}{s}$ and $C(s) = R(s)G(s)$. Therefore,

$$C(s) = \frac{1}{s}\frac{\prod_{i=1}^{r}(s+z_i)}{\prod_{i=1}^{l}(s+p_i)} \tag{17a}$$
$$= \frac{\Gamma(s)}{\Gamma(1+s)}\frac{\prod_{i=1}^{r}\Gamma(z_i+1+s)\prod_{j=1}^{l}\Gamma(p_j+s)}{\prod_{j=1}^{l}\Gamma(p_j+1+s)\prod_{i=1}^{r}\Gamma(z_i+s)} \tag{17b}$$

By using the same reasoning as in the impulse response, it is clear that

$$c(t) = H_{l+r+1,l+r+1}^{l+r+1,0}\left(e^{-t}\left|\begin{array}{l}(1,1),(1+p_j,1)_{1:l},(z_i,1)_{1:r}\\(0,1),(1+z_i,1)_{1:r},(p_j,1)_{1:l}\end{array}\right.\right) \tag{18}$$

2) Ramp Response: Let the input $r(t) = tu(t)$. Therefore, we have $R(s) = \frac{1}{s^2}$ and $C(s) = R(s)G(s)$. Therefore,

$$C(s) = \frac{1}{s^2}\frac{\prod_{i=1}^{r}(s+z_i)}{\prod_{i=1}^{l}(s+p_i)} \tag{19a}$$
$$= \frac{\Gamma(s)\Gamma(s)}{\Gamma(1+s)\Gamma(1+s)}\frac{\prod_{i=1}^{r}\Gamma(z_i+1+s)\prod_{j=1}^{l}\Gamma(p_j+s)}{\prod_{j=1}^{l}\Gamma(p_j+1+s)\prod_{i=1}^{r}\Gamma(z_i+s)} \tag{19b}$$

By using the same reasoning as in the impulse response, it is clear that

$$c(t) = H^{l+r+2,0}_{l+r+2,l+r+2}\left(e^{-t}\middle| \begin{array}{c}(1,1),(1,1),(1+p_j,1)_{1:l},(z_i,1)_{1:r}\\(0,1),(0,1),(1+z_i,1)_{1:r},(p_j,1)_{1:l}\end{array}\right) \tag{20}$$

3) Exponential Response: Let the input $r(t) = e^{-at}u(t)$. Therefore, we have $R(s) = \frac{1}{s+a}$ and $C(s) = R(s)G(s)$. Therefore,

$$C(s) = \frac{1}{s+a}\frac{\prod_{i=1}^{r}(s+z_i)}{\prod_{i=1}^{l}(s+p_i)} \tag{21a}$$

$$= \frac{\Gamma(a+s)}{\Gamma(1+a+s)}\frac{\prod_{i=1}^{r}\Gamma(z_i+1+s)\prod_{j=1}^{l}\Gamma(p_j+s)}{\prod_{j=1}^{l}\Gamma(p_j+1+s)\prod_{i=1}^{r}\Gamma(z_i+s)} \tag{21b}$$

By using the same reasoning as in the impulse response, it is clear that

$$c(t) = H^{l+r+1,0}_{l+r+1,l+r+1}\left(e^{-t}\middle| \begin{array}{c}(1+a,1),(1+p_j,1)_{1:l},(z_i,1)_{1:r}\\(a,1),(1+z_i,1)_{1:r},(p_j,1)_{1:l}\end{array}\right) \tag{22}$$

4) Sinusoidal Response: Let the input $r(t) = \sin(at)u(t)$. Therefore, we have $R(s) = \frac{a}{s^2+a^2}$ and $C(s) = R(s)G(s)$. Therefore,

$$C(s) = \frac{a}{s^2+a^2}\frac{\prod_{i=1}^{r}(s+z_i)}{\prod_{i=1}^{l}(s+p_i)} \tag{23a}$$

$$= \frac{a}{(s+ia)(s-ia)}\frac{\prod_{i=1}^{r}(s+z_i)}{\prod_{i=1}^{l}(s+p_i)} \tag{23b}$$

$$= \frac{a\Gamma(ia+s)\Gamma(-ia+s)}{\Gamma(1+ia+s)\Gamma(1-ia+s)}\frac{\prod_{i=1}^{r}\Gamma(z_i+1+s)\prod_{j=1}^{l}\Gamma(p_j+s)}{\prod_{j=1}^{l}\Gamma(p_j+1+s)\prod_{i=1}^{r}\Gamma(z_i+s)} \tag{23c}$$

By using the same reasoning as in the impulse response, it is clear that

$$c(t) = aH^{l+r+2,0}_{l+r+2,l+r+2}\left(e^{-t}\middle| \begin{array}{c}(1+ia,1),(1-ia,1),(1+p_j,1)_{1:l},(z_i,1)_{1:r}\\(ia,1),(-ia,1),(1+z_i,1)_{1:r},(p_j,1)_{1:l}\end{array}\right) \tag{24}$$

IV. Summary of Results

In summary, we consider an LTI system with the transfer function,

$$G(s) = \frac{C(s)}{R(s)} = \frac{(s+z_1)(s+z_2)...(s+z_r)}{(s+p_1)(s+p_2)...(s+p_l)} \tag{25}$$

where $-z_i, -p_k$ are the poles of $G(s)$. Then, the responses to test functions for the system are given as follows:

A. Impulse Response

Let $r(t) = \delta(t)$ where $r(t)$ is the input in time domain. Then, the output $c(t)$ is given by:

$$c(t) = H^{l+r,0}_{l+r,l+r}\left(e^{-t}\middle| \begin{array}{c}(1+p_j,1)_{1:l},(z_i,1)_{1:r}\\(1+z_i,1)_{1:r},(p_j,1)_{1:l}\end{array}\right) \tag{26}$$

B. Step Response

Let $r(t) = u(t)$ where $r(t)$ is the input in time domain. Then, the output $c(t)$ is given by:

$$c(t) = H^{l+r+1,0}_{l+r+1,l+r+1}\left(e^{-t}\middle|\begin{array}{c}(1,1),(1+p_j,1)_{1:l},(z_i,1)_{1:r}\\(0,1),(1+z_i,1)_{1:r},(p_j,1)_{1:l}\end{array}\right) \tag{27}$$

C. Ramp Response

Let $r(t) = tu(t)$ where $r(t)$ is the input in time domain. Then, the output $c(t)$ is given by:

$$c(t) = H^{l+r+2,0}_{l+r+2,l+r+2}\left(e^{-t}\middle|\begin{array}{c}(1,1),(1,1),(1+p_j,1)_{1:l},(z_i,1)_{1:r}\\(0,1),(0,1),(1+z_i,1)_{1:r},(p_j,1)_{1:l}\end{array}\right) \tag{28}$$

D. Exponential Response

Let $r(t) = e^{-at}u(t)$ where $r(t)$ is the input in time domain. Then, the output $c(t)$ is given by:

$$c(t) = H^{l+r+1,0}_{l+r+1,l+r+1}\left(e^{-t}\middle|\begin{array}{c}(1+a,1),(1+p_j,1)_{1:l},(z_i,1)_{1:r}\\(a,1),(1+z_i,1)_{1:r},(p_j,1)_{1:l}\end{array}\right) \tag{29}$$

E. Sinusoidal Response

Let $r(t) = sin(at)u(t)$ where $r(t)$ is the input in time domain. Then, the output $c(t)$ is given by:

$$c(t) = aH^{l+r+2,0}_{l+r+2,l+r+2}\left(e^{-t}\middle|\begin{array}{c}(1+ia,1),(1-ia,1),(1+p_j,1)_{1:l},(z_i,1)_{1:r}\\(ia,1),(-ia,1),(1+z_i,1)_{1:r},(p_j,1)_{1:l}\end{array}\right) \tag{30}$$

V. Examples

The following section provides examples of system transfer functions, where the responses are analytically plotted using the Fox H-Function, and are compared with the impulse response obtained by numerical Laplace inversion of the transfer function. **In all the figures, the continuous curve is the Fox H-Function, while the superimposed markers are the sampled inverse Laplace transform of the transfer function**. Although SIMULINK results are also in full agreement with the below plots, they have not been included in this report, but will be done so soon.

A. Impulse Response

The system transfer function is:

$$G(s) = \frac{1}{(s+1)(s+2)(s+0.4+1.5j)(s+0.4-1.5j)(s+0.1+2.2j)(s+0.1-2.2j)(s+1+0.4j)(s+1-0.4j)} \tag{31}$$

We now plot the impulse response analytically and numerically. Since the system has no zeros and 8 poles, it is classified as an eighth order LTI system. Using Equation 16, we know immediately that the response of this system to an impulse is given by

$$c(t) = H^{l+r,0}_{l+r,l+r}\left(e^{-t}\middle|\begin{array}{c}(1+p_j,1)_{1:l},(z_i,1)_{1:r}\\(1+z_i,1)_{1:r},(p_j,1)_{1:l}\end{array}\right) \tag{32}$$

Substituting for the values, we have

$$c(t) = H_{0,8}^{8,0}\left(e^{-t}\left|\begin{matrix}(2,1),(3,1),(1.4{+}1.5j,1),(1.4{-}1.5j,1),(1.1+2.2j,1),(1.1-2.2j,1),(2{+}0.4j,1),(2{-}0.4j,1)\\(1,1),(2,1),(0.4{+}1.5j,1),(0.4{-}1.5j,1),(0.1+2.2j,1),(0.1-2.2j,1),(1{+}0.4j,1),(1{-}0.4j,1)\end{matrix}\right.\right) \tag{33}$$

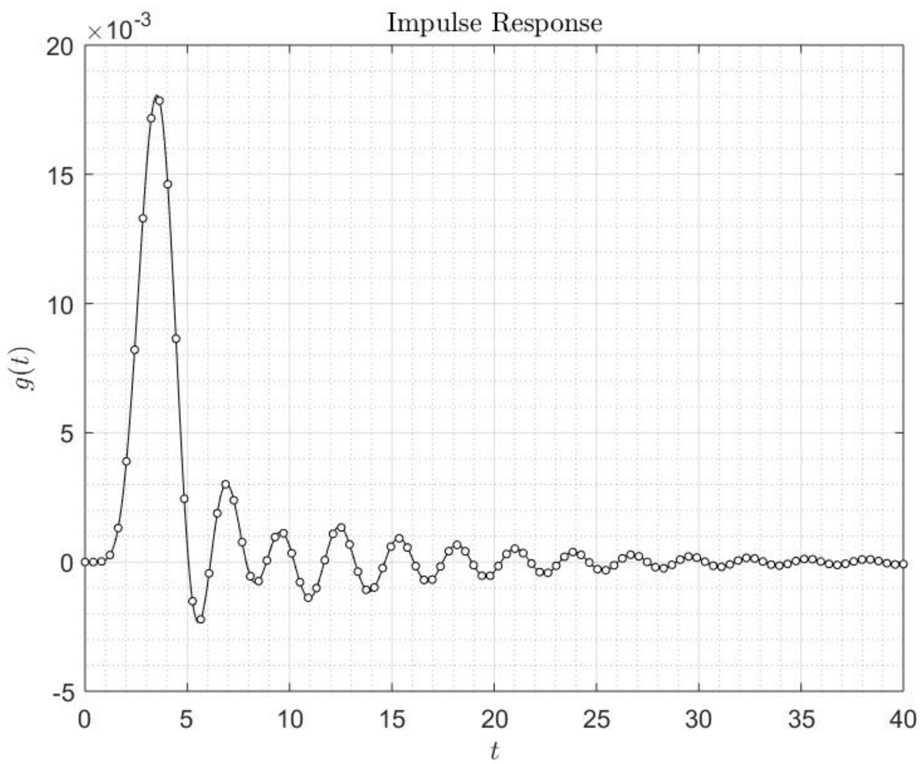

Figure 1. Impulse Response

B. Unit Step Response

The system transfer function is:

$$G(s) = \frac{(s+1)(s+1)(s+5)}{(s+0.5)(s+2)(s+0.4+1.5j)(s+0.4-1.5j)(1+0.5j)(1-0.5j)} \tag{34}$$

We now plot the unit step response analytically and numerically.

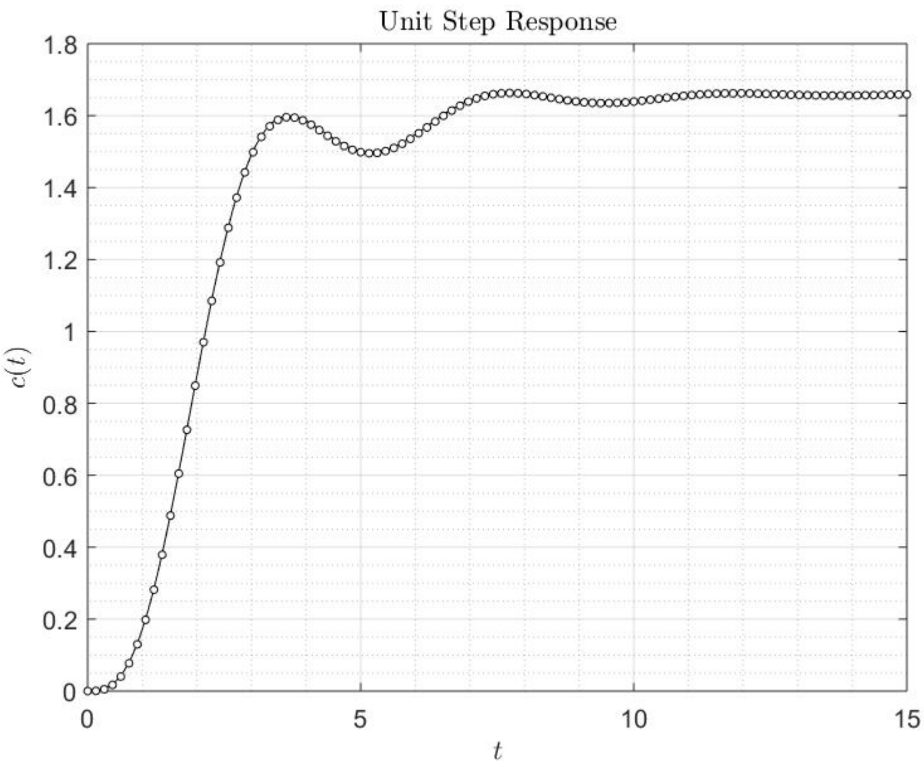

Figure 2. Unit Step Response

C. *Unit Ramp Response*

The system transfer function is:

$$G(s) = \frac{(s+1)(s+2)}{(s+2)(s+0.4+2.5j)(s+0.4-2.5j)} \tag{35}$$

We now plot the unit unit ramp response analytically and numerically.

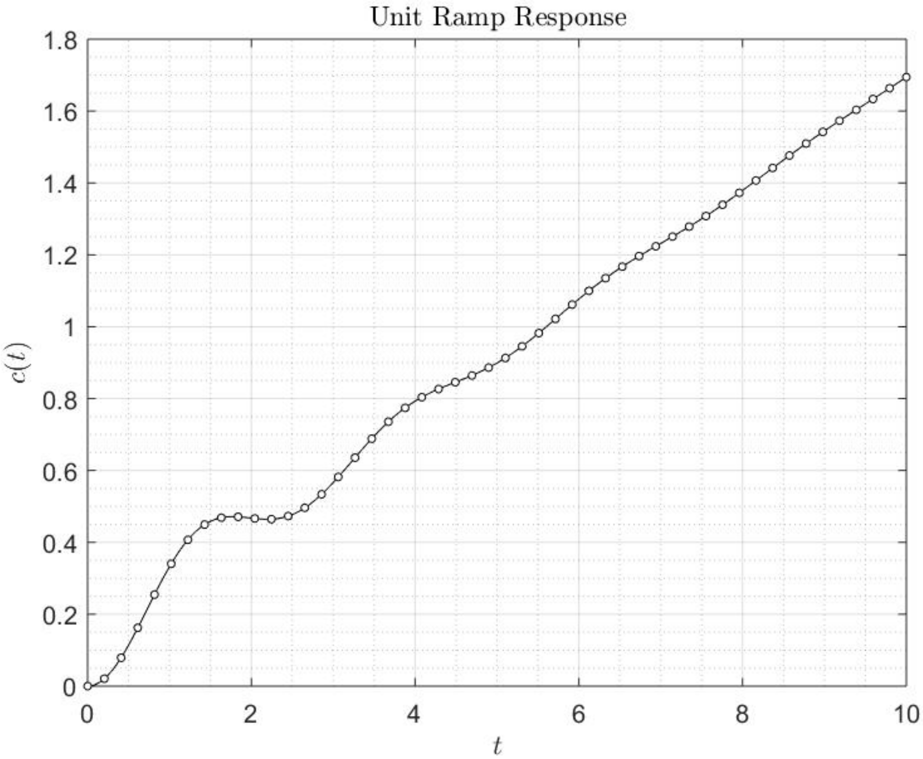

Figure 3. Unit Ramp Response

D. *Unit Sine Response*

The system transfer function is:

$$G(s) = \frac{(s+5)(s+3)(s+2)}{(s+4)(s+0.4+2.5j)(s+0.4-2.5j)(s+3j)(s-3j)} \tag{36}$$

Let the input be $r(t) = \sin(2t)$. Therefore,

$$R(s) = \frac{2}{s^2+4} \tag{37}$$

Using 24, we know that

$$c(t) = 2H_{10,10}^{10,0}\left(e^{-t}\middle|\begin{array}{c}(1+i2,1),(1-i2,1),(1+p_j,1)_{1:l},(z_i,1)_{1:r}\\(i2,1),(-i2,1),(1+z_i,1)_{1:r},(p_j,1)_{1:l}\end{array}\right) \tag{38}$$

This is plotted and compared with numerical inverse Laplace transform.

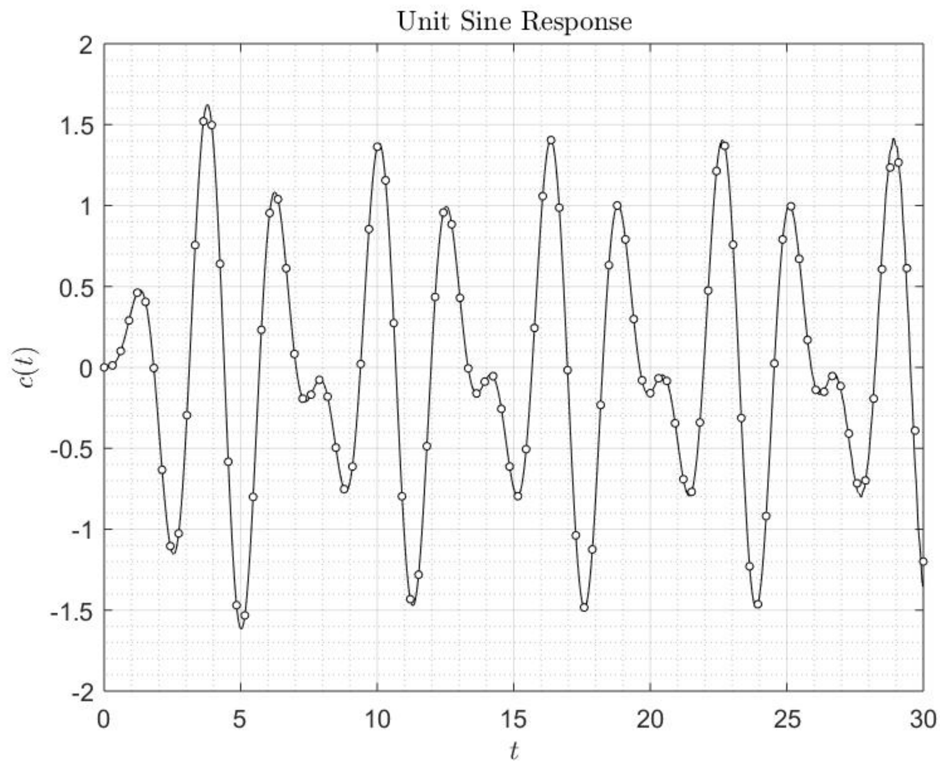

Figure 4. Unit Sine Response

E. Unit Exponential Response

The system transfer function is:

$$G(s) = \frac{(s+1)(s+2)}{(s+2)(s+1+1.2j)(s+1-1.2j)(s+0.1-2j)(s+0.1+2j)} \tag{39}$$

Let the input be $r(t) = e^{-5t}$. Therefore,

$$R(s) = \frac{1}{s+5} \tag{40}$$

Using 22, we know that

$$c(t) = H_{8,8}^{8,0}\left(e^{-t}\middle|\begin{array}{l}(6,1),(1+p_j,1)_{1:l},(z_i,1)_{1:r}\\(5,1),(1+z_i,1)_{1:r},(p_j,1)_{1:l}\end{array}\right) \tag{41}$$

This is plotted and compared with numerical inverse Laplace transform.

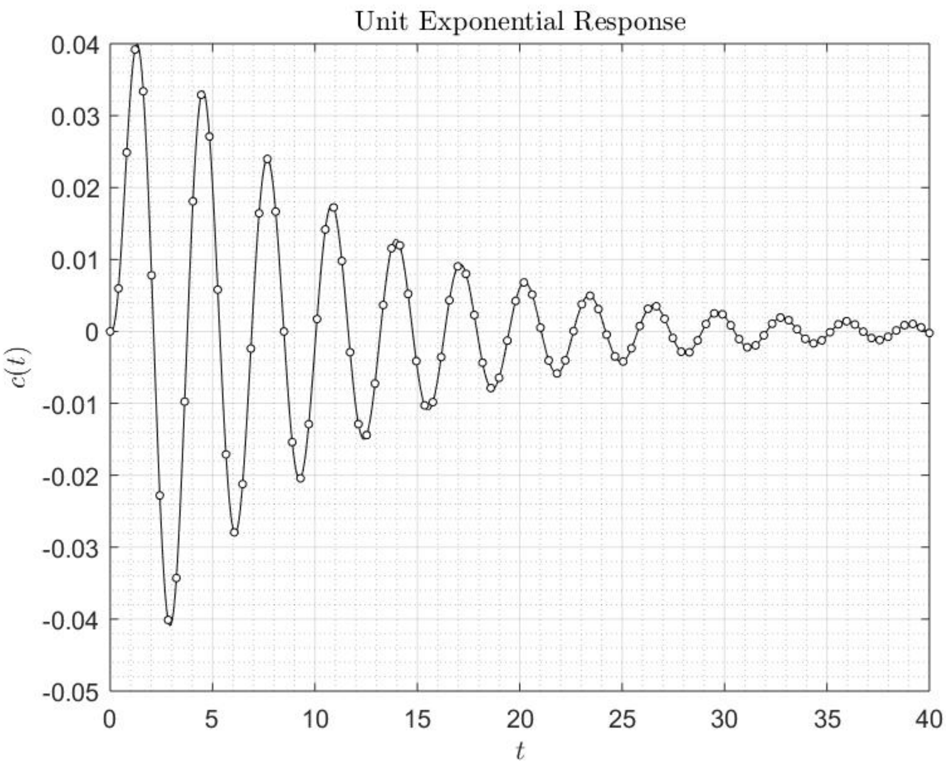

Figure 5. Unit Exponential Response

VI. Algorithms

This section presents the MATLAB algorithm that was implemented in order to plot and obtain the results presented in Section V.

A. Step Response

The following is an algorithm to use the Fox h-Function to obtain the unit step response for a given transfer function. It only requires the poles and zeros as inputs, and automatically generates the parameters for the H-Function, as well as the parameter counts. It then plots the H-Function with respect to time.

Further, it also employs the numerical Inverse Laplace Transform to compare the H-Function with the actual response. The code for generating the discretized and sampled inverse Laplace transform, is provided in the next subsection.

Listing 1: MATLAB Code: Step Response.m

```
%% Zeros and Poles of Transfer Function
zer = [1,1,5];
pol = [0.5,2,0.4+1.5j,0.4-1.5j, 1+0.5j,1-0.5j];
pol = horzcat([0],pol); % Equivalent to Multiplying by 1/s
```

```
%% Parameter Counts
m = length(zer)+length(pol);
n = 0;  % n counts number of gamma functions with a arguments in numerator
p = m;  % p counts number of gamma functions with a arguments in total
q = m;  % q counts number of gamma functions with b arguments in total

%% Parameter Sequence
a = horzcat(1 + pol,zer); % Elements of this set are a_i
A = ones(1,m);            % Elements of this set are A_i
b = horzcat(1 + zer,pol); % Elements of this set are b_j
B = ones(1,m);            % Elements of this set are B_j

%% Scale Parameters
lambda = 1;    % Coefficient of x in argument of H Function
k = 1;         % Normalization factor for H distributions

%% Defining the Integration Parameters
t_final = 15;  % Final Time to Plot
Contour = 0.1; % Point of Intersection of Vertical Contour and Real Axis
step = 0.1;    % Discrete Step Size for Numerical Integration
R = 45;        % Vertical Range of Integration
s1 = Contour+(-1*R:step:R)*1j;   %Vertical Contour

x = 0:0.01:t_final; % Domain of Plotting the H-Function

%% Initializing the Integrand of the H-Function
H1 = zeros(1,length(x));   % Initializing the Output

Sm = ones(1,length(s1));   % Initializing the product of gammas counted by m
Sn = Sm; Sp = Sn; Sq = Sp; % Initializing the remaining product of gammas

%% Constructing the Integrand of the H-Function
for i = 1:m
    Sm = Sm.*Cgamma(b(i)+B(i)*s1);
end
for i = 1:n
    Sn = Sn.*Cgamma(1-a(i)-A(i)*s1);
end
for i = n+1:p
    Sp = Sp.*Cgamma(a(i)+A(i)*s1);
end
```

```
for i = m+1:q
    Sq = Sq.*Cgamma(1-b(i)-B(i)*s1);
end

S = Sm.*Sn./(Sp.*Sq); % Integrand of the H-Function

%% Performing the Numerical Integration
for r = 1:length(x)
        H1(r) = real(sum(S.*(((lambda*(exp(-1*x(r)))).^(-s1))))*step/(2*pi));
end
H = k*H1;  % Scaling Factor for Distributions, k = 1 otherwise.

%%  Obtaining Numerical Inverse Laplace Transform
%   Creating the Transfer Function as a string from zeros and poles
Num = '1';
Den = '1';
for i = 1:length(zer)
    Num = strcat(Num,['*(s+' num2str(zer(i)) ')']);
end
for i = 1:length(pol)
    Den = strcat(Den,['*(s+' num2str(pol(i)) ')']);
end
Den = strcat('(',strcat(Den,')'));
TF = strcat(Num,strcat('/',Den)); % Transfer Function as a String

%%  Numerical Inversion of Laplace Transform
[t1,ft1] = INVLAP(TF,0.001,t_final,1000); % ~Continuous
[t2,ft2] = INVLAP(TF,0.001,t_final,100);  % Discrete

%% Plotting the H-Function
%% Plot H Function and Inverse Laplace Separately
figure(1);
subplot(1,2,1);
plot(x,H);
grid on; grid MINOR;
xlabel('$t$','interpreter','latex');
ylabel(['$H_{' num2str(p) ',' num2str(q) '}^{' num2str(m) ',' num2str(n) '}(e^{-t})$'],'interpreter','latex');
title('Fox H-Function', 'interpreter','latex');

subplot(1,2,2)
plot(t1,ft1);
grid on; grid MINOR;
xlabel('$t$','interpreter','latex');
ylabel(['$\mathcal{L}^{-1} \{ C(s)\}$'],'interpreter','latex');
```

```
title('Numerical Inverse Laplace', 'interpreter','latex');

%% Plot H Function and Inverse Laplace Sueprimposed
figure(2);
hold off;
plot(x,H,'k'); % Plots the H Function
hold on;
plot(t2,ft2,'ko','MarkerEdgeColor','k','MarkerFaceColor','w','MarkerSize',3); % Plots the Inverse Laplace
grid on; grid MINOR;
xlabel('$t$','interpreter','latex');
ylabel('$c(t)$','interpreter','latex');
title('Unit Step Response', 'interpreter','latex');
```

B. Inverse Laplace Transform

The following is a numerical inversion technique, present on the mathworks.com website. It has been modified to meet the requirements of this project.

Listing 2: MATLAB Code: Inverse Laplace Transform.m

```
% INVLAP   Numerical Inversion of Laplace Transforms
function [radt,ft] = INVLAP(Fs,t1,t2,Ns,a,ns,nd)
% Fs is the input function, F(s) as a string.
% t1 and t2 are limits of the solution interval
% Ns is the total number of time instants (samples)

% a, ns, nd are parameters of the method, implicitly defined as
% a = 6, ns = 20, nd = 19 unless otherwise inputted.
% Increasing ns and nd mimizes error but increases complexity.

% Example of Function Call:
% [t,ft]=INVLAP('s/(s^2+4*pi^2)',0,10,1001);
% plot(t,ft)

FF = strrep(strrep(strrep(Fs,'*','.*'),'/','./'),'^','.^');
if nargin == 4
  a  = 6;  % Default Parameter 'a'
  ns = 20; % Default Parameter 'ns'
  nd = 19; % Default Parameter 'nd'
end
radt = linspace(t1,t2,Ns); % Initialize Vector of Time
if t1==0
   radt = radt(2:1:Ns); % t=0 is not allowed
end
tic        % Start Measure CPU Time
```

```
for n = 1:ns+1+nd % Loop for Calculation
   alpha(n)  = a+(n-1)*pi*1j;
   beta(n)   = -exp(a)*(-1)^n;
end
n = 1:nd;
bdif = fliplr(cumsum(gamma(nd+1)./gamma(nd+2-n)./gamma(n)))./2^nd;
beta(ns+2:ns+1+nd)=beta(ns+2:ns+1+nd).*bdif;
beta(1) = beta(1)/2;
for kt=1:Ns  % Cycle for time t
   tt  = radt(kt);
   s   = alpha/tt;        % Complex Frequency, s
   bt  = beta/tt;         % Intermediate Function Value
   btF = bt.*eval(FF);    % Functional value F(s)
   ft(kt)=sum(real(btF)); % Original f(tt)
end
toc
```

VII. Conclusion and Future Work

This section concludes the CTE document, and sheds light on the targets of the project. In this document, a review of the Fox H-Functiona s well as its relationship to integral transforms is provided. This information is used in deriving the impulse response of an arbitrary transfer function, which is disseminated in detail. Following this, similar procedures were used in deriving the response for a variety of test functions. All the results were tested numerically and agreement between the analytical and numerical results is observed.

Looking forward, I aim to utilize the Fox H-Function and the results from this document to analyze non-linear systems, and/or infer their response from the state space domain. Ultimately, it is hoped that the novelty of the approach earns it a publication in a reputed journal.