AMERICAN UNIVERSITY OF SHARJAH

ELE 353: CONTROL SYSTEMS I

PROJECT: FINAL REPORT

The Fox H-Function in Control Systems

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The Fox H-Function in Control Systems

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***Abstract*—This report extends the theory of hypergeometric functions by applying it to Linear and Time Invariant (LTI) systems. It utilizes the univariate Fox H-Function of exponential argument to provide an exact, analytical expression for the time domain system response of LTI systems to a variety of test inputs. These include the step response, impulse response, sine response, ramp response and exponential response. The report introduces the mathematical theory underlying the approach and derives expressions for the responses. It tests the accuracy of these results by comparing the H-Function with inverse laplace transform, and SIMULINK results.**

I. Introduction

Hypergeometric functions are versatile functions that subsume a large category of other functions. An interesting example of such a function is the Fox H-Function, which finds applications in a variety of areas including integrodifferential equations, fractional calculus, astrophysics and probability theory. This project aims to extend its utility by representing the transfer functions and system responses of LTI systems to different test functions, as a single Fox H-Function. The rest of this paper is organized as follows. Section II formulates the problem and provides the mathematical background. Section III derives the system responses based on this background, and Section IV presents the analytical results of the obtained expression and compares it with simulation results. Finally, Section V concludes the paper.

II. Mathematical Background

This section presents the mathematical background and existing results that are utilized in the paper.

A. The Fox H-Function

The Fox H-Function of a scalar complex variable z, henceforth referred to as the H-Function is defined in terms of a Mellin-Barnes integral as shown in Equation 1 [1].

$$H_{p,q}^{m,n}\left(z\left|\begin{matrix}(a_1,A_1),\dots,(a_p,A_p)\\(b_1,B_1),\dots,(b_q,B_q)\end{matrix}\right.\right)=H_{p,q}^{m,n}\left(z\left|\begin{matrix}(a_p,A_p)\\(b_q,B_q)\end{matrix}\right.\right)$$
$$=\frac{1}{2\pi w}\int_L\frac{\prod_{j=1}^{m}\Gamma(b_j+B_js)\prod_{i=1}^{n}\Gamma(1-a_i-A_is)}{\prod_{i=n+1}^{p}\Gamma(a_i+A_is)\prod_{j=m+1}^{q}\Gamma(1-b_j-B_js)}z^{-s}ds \tag{1}$$

Here, $\{m,n,p,q\}\subset\mathbb{N}$ with $0\le n\le p$, $1\le m\le q$, $A_i,B_j\in\mathbb{R}^+$, $a_i,b_j\in\mathbb{C}\ \forall i,j$; $w=\sqrt{-1}$, $z\neq 0$, and $z^{-s}=\exp\{-s(\ln|z|+i\arg z)\}$ where $|\cdot|$ is the absolute value operator and $\arg z$ is not necessarily the principal value. L is a contour on the complex s plane [2] that runs from $c_1-i\infty$ to $c_2+i\infty$, $c_1,c_2\in\mathbb{R}$ such that it separates the poles of the $\Gamma(b_j+B_js)$ terms in the numerator from those of the $\Gamma(1-a_i-A_is)$ terms in the numerator [1]. Hence, L must separate

$$\zeta_{jv}=-\left(\frac{b_j+v}{B_j}\right),\ j\in[1,m]\cap\mathbb{Z};\ v\in\mathbb{N}_0 \tag{2}$$

from

$$\omega_{\lambda k}=\left(\frac{1-a_\lambda+k}{A_\lambda}\right),\ \lambda\in[1,m]\cap\mathbb{Z};\ k\in\mathbb{N}_0 \tag{3}$$

where $\cap$ is the set intersection operator and $\mathbb{N}_0$ is the set of all postive integers including zero.

B. Integral Transforms

This section presents the definitions of the integral transforms used in this paper.

1) Mellin Transform: The Mellin Transform is an integral transform defined as follows. Let $f(t)$ represent a real valued function on $(0,\infty)$, which is essentially the non-transformed space (time domain). The Mellin Transform of $f(t)$ is given as:

$$F(s)=\mathcal{M}\{f(t)\}=\int_0^\infty f(t)t^{s-1}dt \tag{4}$$

This transform is invertible, and the Inverse Mellin Transform of $F(s)$ is given as:

$$f(t)=\mathcal{M}^{-1}\{F(s)\}=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}F(s)t^{-s}ds \tag{5}$$

2) Laplace Transform: The Laplace Transform is an integral transform defined as follows. Let $f(t)$ represent a real valued function on $(-\infty,\infty)$. The Bilateral Laplace Transform of $f(t)$ is given as:

$$F(s)=\mathcal{B}\{f(t)\}=\int_{-\infty}^{\infty}f(t)e^{-st}dt \tag{6}$$

However, Bilateral Transforms do not respect causality, and control systems employ the Unilateral Laplace Transform, defined as

$$F(s)=\mathcal{L}\{f(t)\}=\int_0^\infty f(t)e^{-st}dt \tag{7}$$

This transform is invertible, and the Inverse Laplace Transform of $F(s)$ is given as:

$$f(t)=\mathcal{L}^{-1}\{F(s)\}=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}F(s)e^{st}ds \tag{8}$$

C. Relationships between Integral Transforms

This section highlights the important relationships between integral trasnforms that are utilized in this paper.

1) Relationship between Unilateral and Bilateral Laplace Transform: It is evident that $\mathcal{L}\{f(t)\} = \mathcal{B}\{f(t)u(t)\}$ where $u(t)$ is the Heaviside unit step function. This can be easily proven as follows:

$$\begin{aligned}\mathcal{B}\{f(t)u(t)\} &= \int_{-\infty}^{\infty} f(t)u(t)e^{-st}dt \\ &= \int_{-\infty}^{0^-} f(t)u(t)e^{-st}dt + \int_{0^+}^{\infty} f(t)u(t)e^{-st}dt \\ &= \int_{-\infty}^{0^-} f(t)(0)e^{-st}dt + \int_{0^+}^{\infty} f(t)(1)e^{-st}dt \\ &= \mathcal{L}\{f(t)\} \qquad (9)\end{aligned}$$

This integral converges as countably many finite discontinuities do not affect the Riemann integral value.

2) Relationship between Bilateral Laplace Transform and Mellin Transform: The Bilateral Laplace Transform and the Mellin Transform are related as:

$$\mathcal{B}\{f(t)\} = \mathcal{M}\{f(-\log(t))\} \qquad (10a)$$
$$\mathcal{M}\{f(t)\} = \mathcal{B}\{f(e^{-t})\} \qquad (10b)$$

III. Derivations

This section presents the derivation of expressions for LTI system responses to different outputs. Let t represent the variable time in the non-transformed space. Let s be the transformed variable in Laplace domain. In this discussion, an LTI system is understood to have the following transfer function, with r zeros and l poles including multiplicity:

$$G(s) = \frac{C(s)}{R(s)} = \frac{(s+z_1)(s+z_2)...(s+z_r)}{(s+p_1)(s+p_2)...(s+p_l)} \qquad (11)$$

Here, $C(s)$ and $R(s)$ represent the system output and input in the Laplace Domain, while $-z_i, -p_j \in \mathbb{C}$ are the zeros and poles of the transfer function respectively.

A. Impulse Response

The impulse response of a system with transfer function as in 11 is given by

$$g(t) = \mathcal{L}^{-1}\{G(s)\} \qquad (12)$$

In order to write the impulse response as an H-Function, we define another function, $f(\cdot)$ as follows:

$$f(x) := g(-\log(x)) \qquad (13a)$$
$$g(x) := f(e^{-x}) \qquad (13b)$$

This motivates the following:

$$\begin{aligned}G(s) &= \mathcal{L}\{g(t)\} &(14a)\\ &= \mathcal{B}\{g(t)u(t)\} &(14b)\\ &= \mathcal{B}\{f(e^{-t})u(e^{-t})\} &(14c)\\ &= \mathcal{B}\{f(e^{-t})\} &(14d)\\ &= \mathcal{M}\{f(t)\} &(14e)\\ & &(14f)\end{aligned}$$

Therefore, we infer

$$\begin{aligned}f(t) &= \mathcal{M}^{-1}\{G(s)\} &(15a)\\ &= \mathcal{M}^{-1}\left\{\frac{(s+z_1)(s+z_2)...(s+z_r)}{(s+p_1)(s+p_2)...(s+p_l)}\right\} &(15b)\\ &= \mathcal{M}^{-1}\left\{\frac{\prod_{i=1}^{r}(s+z_i)}{\prod_{i=1}^{l}(s+p_i)}\right\} &(15c)\\ &= \mathcal{M}^{-1}\left\{\frac{\prod_{i=1}^{r}\frac{\Gamma(s+z_i+1)}{\Gamma(s+z_i)}}{\prod_{j=1}^{l}\frac{\Gamma(s+p_j+1)}{\Gamma(s+p_j)}}\right\} &(15d)\\ &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\prod_{i=1}^{r}\Gamma(z_i+1+s)\prod_{j=1}^{l}\Gamma(p_j+s)}{\prod_{j=1}^{l}\Gamma(p_j+1+s)\prod_{i=1}^{r}\Gamma(z_i+s)}t^{-s}ds &(15e)\\ &= H_{l+r,l+r}^{l+r,0}\left(t\,\middle|\,\begin{matrix}(1+p_j,1)_{1:l},(z_i,1)_{1:r}\\(1+z_i,1)_{1:r},(p_j,1)_{1:l}\end{matrix}\right) &(15f)\end{aligned}$$

Finally,

$$\begin{aligned}g(t) &= f(e^{-t})\\ &= H_{l+r,l+r}^{l+r,0}\left(e^{-t}\,\middle|\,\begin{matrix}(1+p_j,1)_{1:l},(z_i,1)_{1:r}\\(1+z_i,1)_{1:r},(p_j,1)_{1:l}\end{matrix}\right) \qquad (16)\end{aligned}$$

This provides an exact, closed-form expression for the impulse response of any LTI system, regardless of the order.

B. Unit Step Response

Let the input $r(t) = u(t)$. Therefore, we have $R(s) = \frac{1}{s}$ and $C(s) = R(s)G(s)$. Therefore,

$$\begin{aligned}C(s) &= \frac{1}{s}\frac{\prod_{i=1}^{r}(s+z_i)}{\prod_{i=1}^{l}(s+p_i)} &(17a)\\ &= \frac{\Gamma(s)}{\Gamma(1+s)}\frac{\prod_{i=1}^{r}\Gamma(z_i+1+s)\prod_{j=1}^{l}\Gamma(p_j+s)}{\prod_{j=1}^{l}\Gamma(p_j+1+s)\prod_{i=1}^{r}\Gamma(z_i+s)} &(17b)\end{aligned}$$

By following a similar line of reasoning as the impulse response, it is clear that

$$c(t) = H_{l+r+1,l+r+1}^{l+r+1,0}\left(e^{-t}\,\middle|\,\begin{matrix}(1,1),(1+p_j,1)_{1:l},(z_i,1)_{1:r}\\(0,1),(1+z_i,1)_{1:r},(p_j,1)_{1:l}\end{matrix}\right) \qquad (18)$$

C. Ramp Response

Let the input $r(t) = tu(t)$. Therefore, we have $R(s) = \frac{1}{s^2}$ and $C(s) = R(s)G(s)$. Therefore,

$$\begin{aligned}C(s) &= \frac{1}{s^2}\frac{\prod_{i=1}^{r}(s+z_i)}{\prod_{i=1}^{l}(s+p_i)} &(19a)\\ &= \frac{\Gamma(s)\Gamma(s)}{\Gamma(1+s)\Gamma(1+s)}\frac{\prod_{i=1}^{r}\Gamma(z_i+1+s)\prod_{j=1}^{l}\Gamma(p_j+s)}{\prod_{j=1}^{l}\Gamma(p_j+1+s)\prod_{i=1}^{r}\Gamma(z_i+s)} &(19b)\end{aligned}$$

By using the same reasoning as in the impulse response, it is clear that

$$c(t) = H_{l+r+2,l+r+2}^{l+r+2,0}\left(e^{-t}\,\middle|\,\begin{matrix}(1,1),(1,1),(1+p_j,1)_{1:l},(z_i,1)_{1:r}\\(0,1),(0,1),(1+z_i,1)_{1:r},(p_j,1)_{1:l}\end{matrix}\right) \qquad (20)$$

D. Exponential Response

Let the input $r(t) = e^{-at}u(t)$. Therefore, we have $R(s) = \frac{1}{s+a}$ and $C(s) = R(s)G(s)$. Therefore,

$$C(s) = \frac{1}{s+a} \frac{\prod_{i=1}^{r}(s+z_i)}{\prod_{i=1}^{l}(s+p_i)} \tag{21a}$$

$$= \frac{\Gamma(a+s)}{\Gamma(1+a+s)} \frac{\prod_{i=1}^{r}\Gamma(z_i+1+s)\prod_{j=1}^{l}\Gamma(p_j+s)}{\prod_{j=1}^{l}\Gamma(p_j+1+s)\prod_{i=1}^{r}\Gamma(z_i+s)} \tag{21b}$$

By using the same reasoning as in the impulse response, it is clear that

$$c(t) = H_{l+r+1,l+r+1}^{l+r+1,0}\left(e^{-t}\middle|\begin{matrix}(1+a,1),(1+p_j,1)_{1:l},(z_i,1)_{1:r}\\(a,1),(1+z_i,1)_{1:r},(p_j,1)_{1:l}\end{matrix}\right) \tag{22}$$

E. Sinusoidal Response

Let the input $r(t) = \sin(at)u(t)$. Therefore, we have $R(s) = \frac{a}{s^2+a^2}$ and $C(s) = R(s)G(s)$. Therefore,

$$C(s) = \frac{a}{s^2+a^2} \frac{\prod_{i=1}^{r}(s+z_i)}{\prod_{i=1}^{l}(s+p_i)} \tag{23a}$$

$$= \frac{a}{(s+ia)(s-ia)} \frac{\prod_{i=1}^{r}(s+z_i)}{\prod_{i=1}^{l}(s+p_i)} \tag{23b}$$

Using the fact that

$$\frac{a}{(s+ia)(s-ia)} = \frac{a\Gamma(ia+s)\Gamma(-ia+s)}{\Gamma(1+ia+s)\Gamma(1-ia+s)}$$

and combining the gamma functions, the sinusoidal response is obtained as provided in 24.

IV. RESULTS

This section presents analytical responses to systems given transfer functions in terms of the H-Function. It plots the response and compares it with the numerical solution obtained by inverting the laplace transform or simulation results obtained by implementing the system in SIMULINK.

A. Impulse Response

Consider the system transfer function given in 25. The impulse response is then plotted analytically and numerically.

Since the system has no zeros and 8 poles, it is classified as an eighth order LTI system. Using Equation 16, we know immediately that the response of this system to an impulse is given by 26. Substituting for the values, we get the expression in 27, which is the impulse response. This H-Function is plotted using MATLAB, and the transfer function is numerically inverted. Both plots are superimposed and presented in Figure 1. Note that in this figure as well as all others, the solid curve is the H-Function, and the circular markers is the sampled inverse laplace transform.

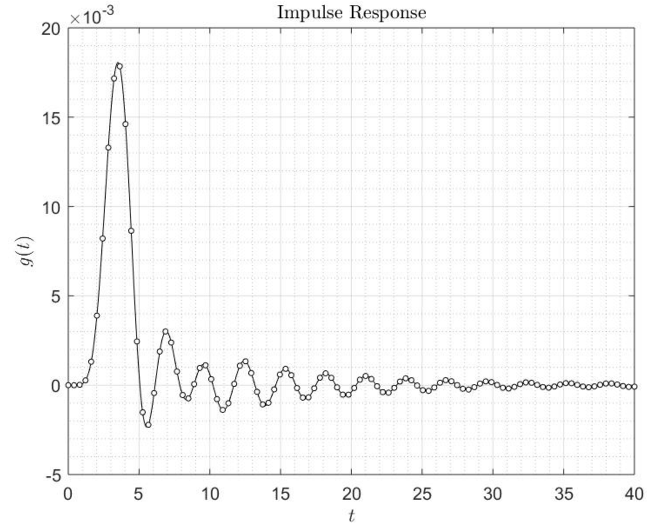

Figure 1. Impulse Response

B. Unit Step Response

Let the system transfer function be as given in 28. Then the step response is given by 29. Therefore, substituting the values gives the expression provided in 30. This step response is plotted analytically and numerically and provided in Figure 2.

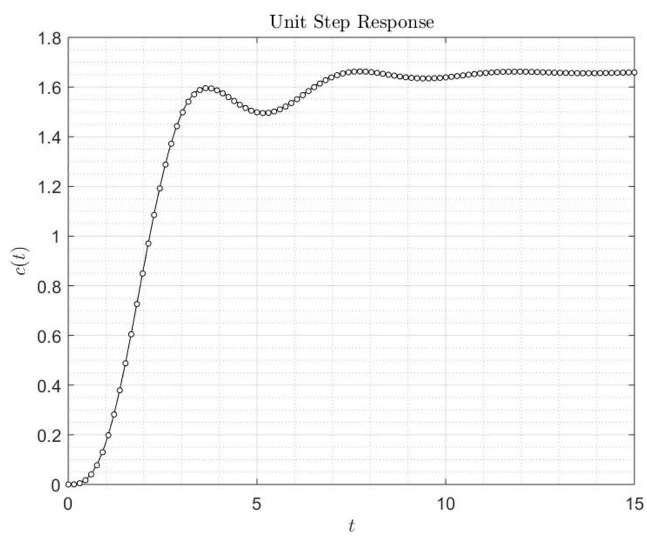

Figure 2. Unit Step Response

C. Unit Ramp Response

Let the system transfer function be as given in 31.

$$G(s) = \frac{(s+1)(s+2)}{(s+2)(s+0.4+2.5j)(s+0.4-2.5j)} \tag{31}$$

Then the step response is given by 20. Therefore, substituting the values gives the expression provided in 32. This step response is plotted analytically and numerically and provided in Figure 3.

$$c(t) = aH^{l+r+2,0}_{l+r+2,l+r+2}\left(e^{-t}\middle|\begin{array}{c}(1+ia,1),(1-ia,1),(1+p_j,1)_{1:l},(z_i,1)_{1:r}\\(ia,1),(-ia,1),(1+z_i,1)_{1:r},(p_j,1)_{1:l}\end{array}\right) \tag{24}$$

$$G(s) = \frac{1}{(s+1)(s+2)(s+0.4+1.5j)(s+0.4-1.5j)(s+0.1+2.2j)(s+0.1-2.2j)(s+1+0.4j)(s+1-0.4j)} \tag{25}$$

$$c(t) = H^{l+r,0}_{l+r,l+r}\left(e^{-t}\middle|\begin{array}{c}(1+p_j,1)_{1:l},(z_i,1)_{1:r}\\(1+z_i,1)_{1:r},(p_j,1)_{1:l}\end{array}\right) \tag{26}$$

$$c(t) = H^{8,0}_{0,8}\left(e^{-t}\middle|\begin{array}{c}(2,1),(3,1),(1.4{+}1.5j,1),(1.4{-}1.5j,1),(1.1{+}2.2j,1),(1.1-2.2j,1),(2{+}0.4j,1),(2{-}0.4j,1)\\(1,1),(2,1),(0.4{+}1.5j,1),(0.4{-}1.5j,1),(0.1{+}2.2j,1),(0.1-2.2j,1),(1{+}0.4j,1),(1{-}0.4j,1)\end{array}\right) \tag{27}$$

$$G(s) = \frac{(s+1)(s+1)(s+5)}{(s+0.5)(s+2)(s+0.4+1.5j)(s+0.4-1.5j)(s+1+0.5j)(s+1-0.5j)} \tag{28}$$

$$c(t) = H^{l+r+1,0}_{l+r+1,l+r+1}\left(e^{-t}\middle|\begin{array}{c}(1,1),(1+p_j,1)_{1:l},(z_i,1)_{1:r}\\(0,1),(1+z_i,1)_{1:r},(p_j,1)_{1:l}\end{array}\right) \tag{29}$$

$$c(t) = H^{10,0}_{10,10}\left(e^{-t}\middle|\begin{array}{c}(1,1),(1.4{+}1.5j,1),(1.4{-}1.5j,1),(2{+}0.5j,1),(2-0.5j,1),(1.5,1),(3,1),(1,1),(1,1),(5,1)\\(0,1),(0.4{+}1.5j,1),(0.4{-}1.5j,1),(1{+}0.5j,1),(1-0.5j,1),(0.5,1),(2,1),(2,1),(2,1),(6,1)\end{array}\right) \tag{30}$$

$$c(t) = H^{7,0}_{7,7}\left(e^{-t}\middle|\begin{array}{c}(1,1),(1,1),(3,1),(1.4+2.5j,1),(1.4-2.5j,1),(1,1),(2,1)\\(0,1),(0,1),(2,1),(3,1),(2,1),(0.4+2.5j),(0.4-2.5j)\end{array}\right) \tag{32}$$

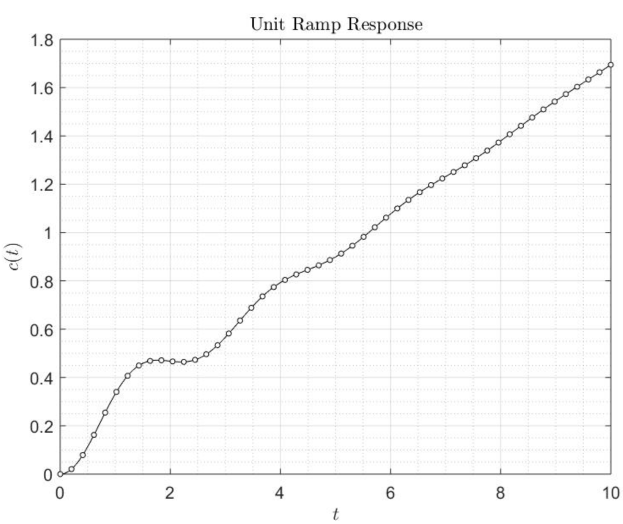

Figure 3. Unit Step Response

D. Unit Sinusoidal Response

Let the transfer function be given by 33.

$$G(s) = \frac{(s+5)(s+3)(s+2)}{(s+4)(s+0.4+2.5j)(s+0.4-2.5j)(s+3j)(s-3j)} \tag{33}$$

Let the input be $r(t) = \sin(2t)$. Then the step response is given by 20. Therefore, substituting the values gives the expression provided in 34. This step response is plotted analytically and numerically and provided in Figure 4.

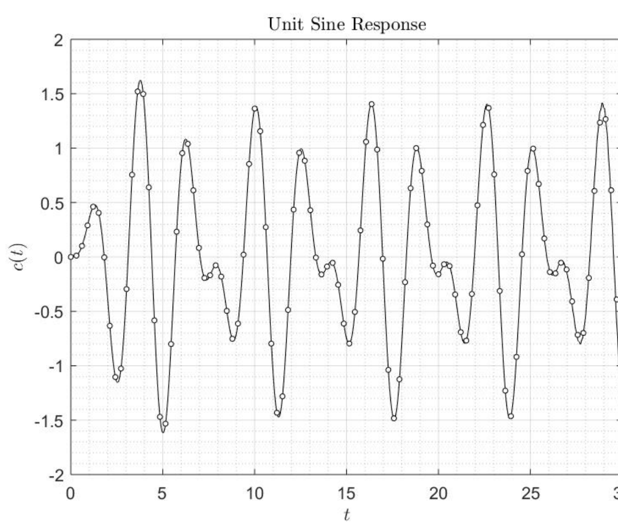

Figure 4. Unit Step Response

E. Unit Exponential Response

Let the system transfer function be given by 35:
Let the input be $r(t) = e^{-5t}$. Using 22, we know that

$$c(t) = H^{8,0}_{8,8}\left(e^{-t}\middle|\begin{array}{c}(6,1),(1+p_j,1)_{1:l},(z_i,1)_{1:r}\\(5,1),(1+z_i,1)_{1:r},(p_j,1)_{1:l}\end{array}\right) \tag{36}$$

$$c(t)=2H_{10,10}^{10,0}\left(e^{-t}\middle|\begin{matrix}(1+2j,1),(1-2j,1),(5,1),(1.4+2.5j,1),(1.4-2.5j,1),(4j,1),(-4j,1),(5,1),(3,1),(2,1)\\(2j,1),(-2j,1),(6,1),(4,1),(3,1),(4,1),(0.4+2.5j,1),(0.4-2.5j,1),(3j,1),(-3j,1)\end{matrix}\right) \tag{34}$$

$$G(s)=\frac{(s+1)(s+2)}{(s+2)(s+1+1.2j)(s+1-1.2j)(s+0.1-2j)(s+0.1+2j)} \tag{35}$$

$$c(t)=H_{8,8}^{8,0}\left(e^{-t}\middle|\begin{matrix}(6,1),(3,1),(2+1.2j,1),(2-1.2j),(1.1-2j),(1.1+2j),(1,1),(2,1)\\(5,1),(2,1),(3,1),(2,1),(1+1.2j,1),(1-1.2j),(0.1-2j),(0.1+2j)\end{matrix}\right) \tag{37}$$

Substituting the values gives the expression for the response as provided in 37. This is plotted and compared with numerical inverse Laplace transform.

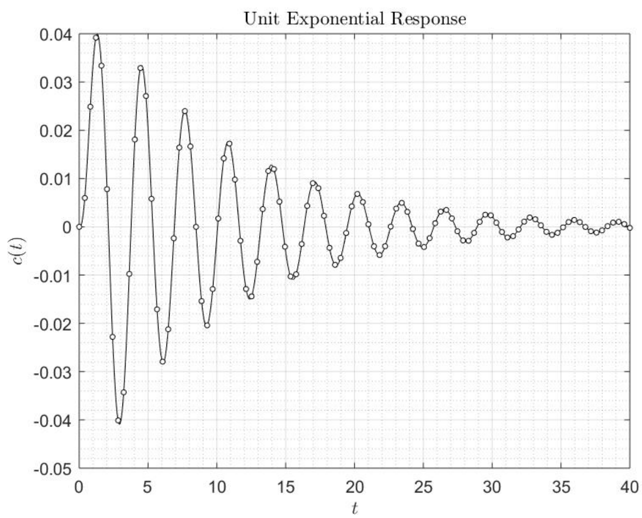

Figure 5. Unit Exponential Response

F. SIMULINK Implementation

1) SIMULINK Implementation 1: Consider the following block diagram implementation in SIMULINK, as shown in Figure 6.

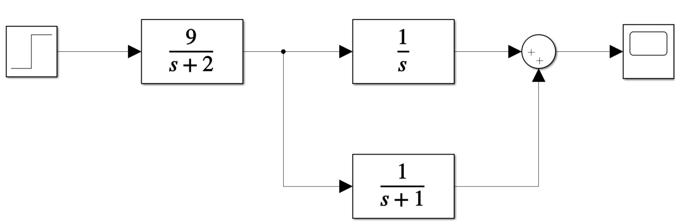

Figure 6. SIMULINK Block Diagram Implementation 1

The result of this implementation is shown in Figure 7, obtained by exporting the scope display from SIMULINK.

The block diagram can be reduced so that the transfer function of the system is given by:

$$C(s)=\frac{18s+9}{s^3+3s^2+2s}=\frac{18(s+0.5)}{s(s+2)(s+1)} \tag{38}$$

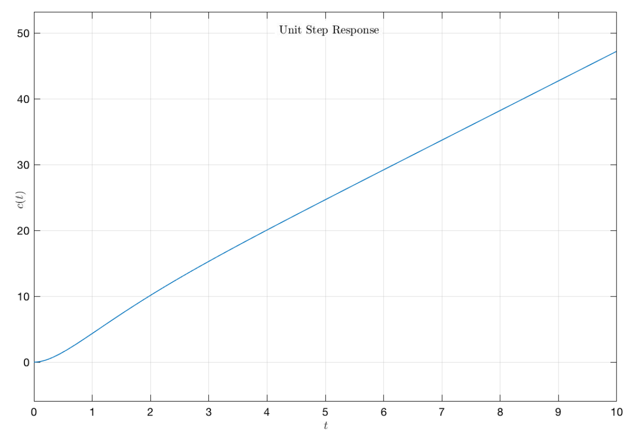

Figure 7. SIMULINK Step Response 1

The transfer function in 38 is used to plot the H-Function. The result is provided in Figure 8

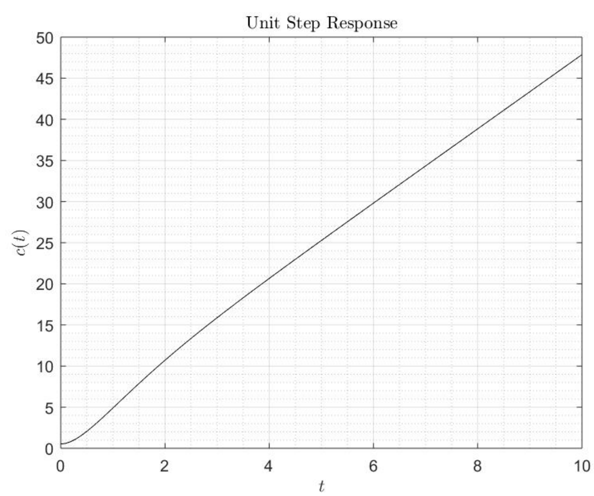

Figure 8. H-Function Step Response 1

It is clear that the results are in agreement.

2) SIMULINK Implementation 2: Consider the following block diagram implementation in SIMULINK, as shown in Figure 9.

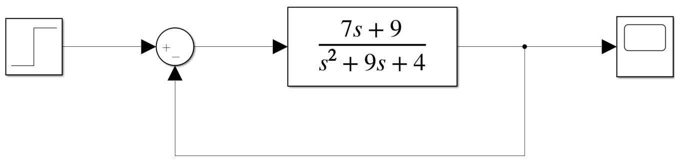

Figure 9. SIMULINK Block Diagram Implementation 2

The result of this implementation is shown in Figure 10, obtained by exporting the scope display from SIMULINK.

The block diagram can be reduced so that the transfer function of the system is given by:

$$C(s) = \frac{7s+9}{s^2+16s+13} = \frac{7(s+1.2857)}{(s+15.1414)(s+0.8586)} \tag{39}$$

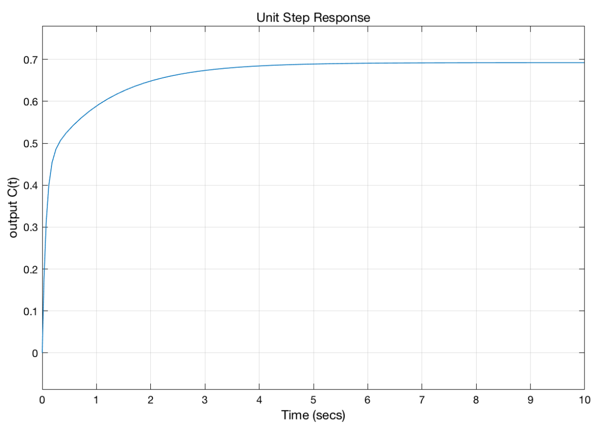

Figure 10. SIMULINK Step Response 1

The transfer function in 38 is used to plot the H-Function. The result is provided in Figure 8

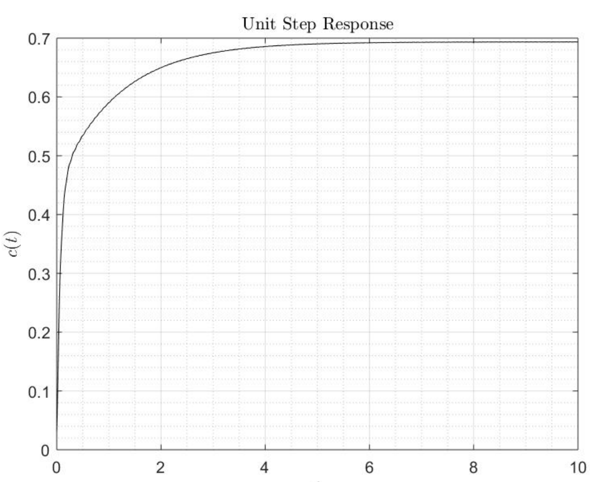

Figure 11. H-Function Step Response 2

It is clear that the MATLAB analytical results are in agreement with the SIMULINK simulations.

V. Conclusion

This paper used the Fox H-Function to derive expressions for LTI system response to various test functions. The expressions are closed-form and exact and are expressed as a single univariate Fox H-Function of exponential argument, in the case of impulse, sine, exponential, ramp and step inputs. The paper showed that these responses can always be expressed as a single H-function, regardless of the order of the system. This proivdes a tool to describe the output of LTI systems in time domain. The derived expressions are verified for their accuracy by comparing the results with numerically inverting the transfer function, as well as with the results from simulation of the system in SIMULINK. In all cases, it is observed that the results are in agreement with the analytical H-Function.

Future work involves extending the approach of analysis to LTV systems, as well as using the H-Function to perform state-space analysis. This would provide a powerful tool to analyze control systems and judge their stability in a manner whose complexity is independent of the order of the system.

References

[1] A. Mathai, R. Saxena, and H. Haubold, *The H-Function: Theory and Applications*. Springer New York, 2009.

[2] D. G. Zill and P. Shanahan, *Complex Analysis*. USA: Jones and Bartlett Publishers, Inc., 3rd ed., 2013.