American University of Sharjah

MTH 351: Methods of Applied Mathematics

Project Report

The Fox H-Function and its Applications

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April 23, 2019

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Abstract

This report is a document that summarizes the results of the project, submitted in partial fulfillment of the requirements for the course MTH 351 - Methods of Applied Mathematics. The report introduces the Fox H-Function by motivating the requirement for hypergeometric functions. It then presents the definition and description of the function, followed by its special cases. These cases are complemented by their utility in the engineering and physical sciences. Furthermore, some properties of the function, particularly with respect to integral transform theory and calculus are discussed and presented. This background is then manifested as applications in the second part of the report, wherein applications of the Fox H-Function with respect to statistics, probability and control theory are presented.

1 Introduction

The theory of hypergeometric functions has gained considerable attention in recent years and found its way into engineering literature. Perhaps the most notable of these functions is Fox's H-Function, introduced in 1961 as a symmetric fourier kernel to Meijer's G-Function. The function has since exploded in popularity and found applications in fractional integro-differential equations, wireless communications, statistical distribution theory and astrophysics among other fields. The justification behind this diversity stems from the mathematical properties that the function confers. Not only does it encompass a wide range of named special functions and statistical distributions with semi-infinite support, it also presents a myriad of properties that allow otherwise complicated mathematical manipulations.

2 Integral Transforms

In this section, some integral transforms that are useful to the development of this project are presented.

2.1 Mellin Transform

The Mellin Transform is an integral transform defined as follows. Let $f(t)$ represent a real valued function on $(0, \infty)$, which is essentially the non-transformed space (time domain). The Mellin Transform of $f(t)$ is given as:

$$F(s) = \mathcal{M}\{f(t)\} = \int_0^\infty f(t)t^{s-1}dt \tag{1}$$

This transform is invertible, and the Inverse Mellin Transform of $F(s)$ is given as:

$$f(t) = \mathcal{M}^{-1}\{F(s)\} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(s)t^{-s}ds \tag{2}$$

Example 2.1. *Let $f(t) = e^{-t}$. Then, the Mellin Transform of $f(t)$ is given as*

$$F(s) = \mathcal{M}\{e^{-t}\} = \int_0^\infty e^{-t}t^{s-1}dt = \Gamma(s) \tag{3}$$

2.2 Laplace Transform

The Laplace Transform is an integral transform defined as follows. Let $f(t)$ represent a real valued function on $(-\infty, \infty)$. The Bilateral Laplace Transform of $f(t)$ is given as:

$$F(s) = \mathcal{B}\{f(t)\} = \int_{-\infty}^{\infty} f(t)e^{-st}dt \tag{4}$$

However, Bilateral Transforms do not respect causality, and control systems employ the Unilateral Laplace Transform, defined as

$$F(s) = \mathcal{L}\{f(t)\} = \int_{0}^{\infty} f(t)e^{-st}dt \tag{5}$$

This transform is invertible, and the Inverse Laplace Transform of $F(s)$ is given as:

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(s)e^{st}ds \tag{6}$$

3 The Fox H-Function

3.1 The Gamma Function of Complex Argument

The Gamma function is defined simply as

$$\Gamma(s) = \int_{0}^{\infty} t^{s-1}e^{-t}dt \tag{7}$$

Here, $s \in \mathbb{C}$ is allowed as the equation can simply be applied to the complex number in concern. Hence, it is possible to write

$$\Gamma(a+ib) = u + iv,$$

where $a, b, u, v \in \mathbb{R}$. This cannot be plotted directly as the domain space as well as codomain space is two dimensional, and hence requires four dimensions to plot. However, we can plot the absolute value of the result, effectively condensing the plot to three dimensions. This is presented in figure 1, where the absolute value of the result is plotted against the real and imaginary components. Slicing the result for when the imaginary part of the input is zero yields the graph of the gamma function over the reals as we know it.

It is interesting to observe that the extension of the domain of the Gamma function to the complex numbers does not introduce any additional poles. Hence, the poles of the Gamma function of complex argument continue to only be the negative integers. This fact proves useful for later analysis.

3.2 Definition of the Fox H-Function

The Fox H-Function of a scalar complex variable z, henceforth referred to as the H-Function is defined in terms of a Mellin-Barnes integral as follows [1]:

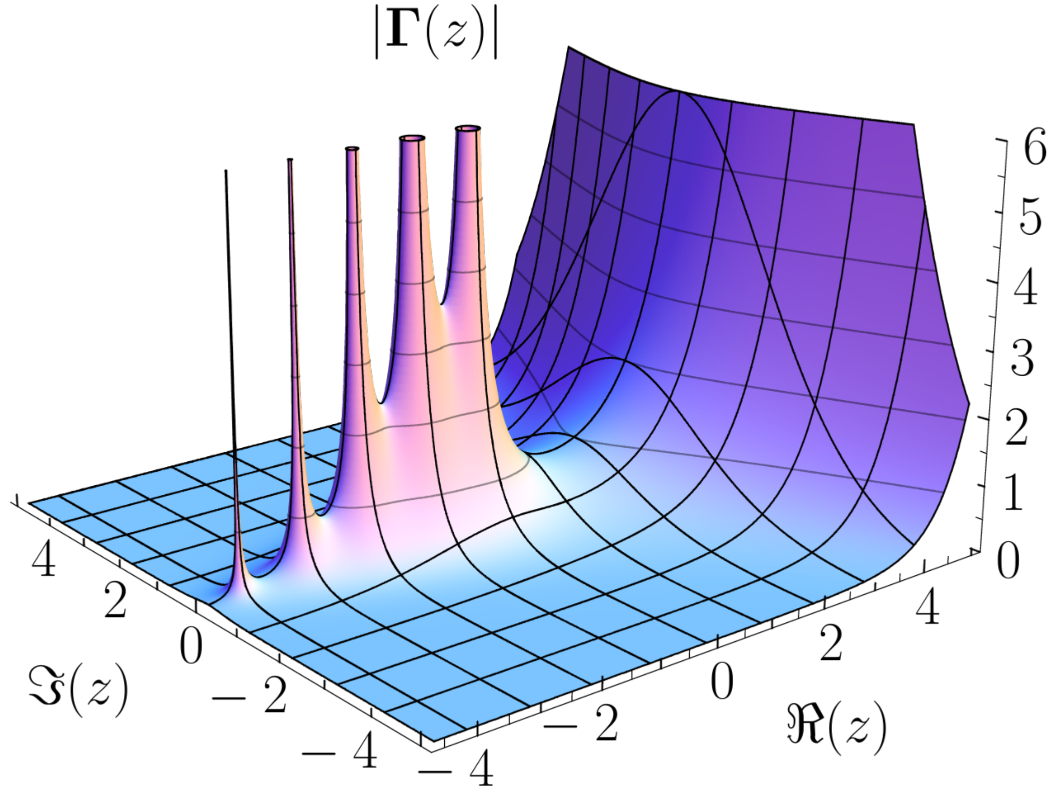

Figure 1: The Gamma Function of Complex Argument

$$H_{p,q}^{m,n}\left(z\left|\begin{array}{c}(a_1,A_1),\ldots,(a_p,A_p)\\(b_1,B_1),\ldots,(b_q,B_q)\end{array}\right.\right)=H_{p,q}^{m,n}\left(z\left|\begin{array}{c}(a_p,A_p)\\(b_q,B_q)\end{array}\right.\right)=H_{p,q}^{m,n}(z)=H(z)$$
$$=\frac{1}{2\pi w}\int_L\frac{\prod_{j=1}^m\Gamma(b_j+B_js)\prod_{i=1}^n\Gamma(1-a_i-A_is)}{\prod_{i=n+1}^p\Gamma(a_i+A_is)\prod_{j=m+1}^q\Gamma(1-b_j-B_js)}z^{-s}ds \quad (8)$$

where $\{m,n,p,q\}\subset\mathbb{N}$ with $0\le n\le p$, $1\le m\le q$, $A_i,B_j\in\mathbb{R}^+$, $a_i,b_j\in\mathbb{C}\ \forall i,j$; $w=\sqrt{-1}$, $z\neq 0$, and $z^{-s}=\exp\{-s(\ln|z|+i\arg z)\}$ where $|\cdot|$ is the absolute value operator and $\arg z$ is not necessarily the principal value. L is a contour on the complex s plane [3] that runs from $c_1-i\infty$ to $c_2+i\infty$, $c_1,c_2\in\mathbb{R}$ such that it separates the poles of the $\Gamma(b_j+B_js)$ terms in the numerator from those of the $\Gamma(1-a_i-A_is)$ terms in the numerator [1]. Hence, L must separate

$$\zeta_{jv}=-\left(\frac{b_j+v}{B_j}\right),\quad j\in[1,m]\cap\mathbb{Z};\ v\in\mathbb{N}_0 \quad (9)$$

from

$$\omega_{\lambda k} = \left(\frac{1 - a_\lambda + k}{A_\lambda} \right), \quad \lambda \in [1, m] \cap \mathbb{Z};\ k \in \mathbb{N}_0 \tag{10}$$

where $\cap$ is the set intersection operator and $\mathbb{N}_0$ is the set of all positive integers including zero.

3.3 Relationship between Mellin Transform and Gamma Function

In this section, we establish the relationship between the Mellin transform and the Gamma Function.

From the definitions, it is clear that:

$$H_{p,q}^{m,n} \left(z \middle| \begin{array}{c} (a_1, A_1), \ldots, (a_p, A_p) \\ (b_1, B_1), \ldots, (b_q, B_q) \end{array} \right) = \mathcal{M}^{-1} \left\{ \frac{\prod_{j=1}^{m} \Gamma(b_j + B_j s) \prod_{i=1}^{n} \Gamma(1 - a_i - A_i s)}{\prod_{i=n+1}^{p} \Gamma(a_i + A_i s) \prod_{j=m+1}^{q} \Gamma(1 - b_j - B_j s)} \right\} \tag{11}$$

Example 3.1. *Set $m = 1 = 1, n = p = 0$ and $(b_1, B_1) = (0, 1)$. Then,*

$$H_{0,1}^{1,0} \left(z \middle| \begin{array}{c} - \\ (0, 1) \end{array} \right) = \mathcal{M}^{-1}\{\Gamma(s)\} = e^{-z} \tag{12}$$

Therefore, the exponential function is a special case of the H-Function.

4 Special Cases and Properties

4.1 Special Cases of the H-Function

4.1.1 Power and Exponential Functions

1.

$$z^b = H_{1,1}^{1,0} \left(z \middle| \begin{array}{c} (b + 1, 1) \\ (b, 1) \end{array} \right)$$

2.

$$\frac{z^b}{(1 + z)^a} = \frac{1}{\Gamma(a)} H_{1,1}^{1,1} \left(z \middle| \begin{array}{c} (b - a + 1, 1) \\ (b, 1) \end{array} \right)$$

3.

$$e^{-z} = H_{0,1}^{1,0} \left(z \middle| \begin{array}{c} - \\ (0, 1) \end{array} \right)$$

4.

$$B^{-1} z^{\frac{b}{B}} e^{-z^{\frac{1}{B}}} = H_{0,1}^{1,0} \left(z \middle| \begin{array}{c} - \\ (b, B) \end{array} \right)$$

4.1.2 Trigonometric Functions

1. Sine Function

$$\sin(z) = \frac{\sqrt{\pi}}{2} H_{0,2}^{1,0}\left(\frac{z}{2} \middle| \begin{matrix} - \\ (\frac{1}{2}, \frac{1}{2}), (0, \frac{1}{2}) \end{matrix} \right)$$

2. Cosine Function

$$\cos(z) = \frac{\sqrt{\pi}}{2} H_{0,2}^{1,0}\left(\frac{z}{2} \middle| \begin{matrix} - \\ (0, \frac{1}{2}), (\frac{1}{2}, \frac{1}{2}) \end{matrix} \right)$$

3. Inverse Sine Function

$$\sin^{-1}(z) = \frac{1}{4} H_{2,2}^{1,2}\left(iz \middle| \begin{matrix} (\frac{1}{2}, \frac{1}{2}), (\frac{1}{2}, \frac{1}{2}) \\ (0, \frac{1}{2}), (\frac{-1}{2}, \frac{1}{2}) \end{matrix} \right)$$

4. Inverse Tangent Function

$$\tan^{-1}(z) = \frac{1}{4} H_{2,2}^{1,2}\left(iz \middle| \begin{matrix} (1, \frac{1}{2}), (\frac{1}{2}, \frac{1}{2}) \\ (\frac{1}{2}, \frac{1}{2}), (0, \frac{1}{2}) \end{matrix} \right)$$

4.1.3 Hyperbolic Functions

1. Hyperbolic Sine Function

$$\sinh(z) = \frac{-\sqrt{\pi} i}{2} H_{0,2}^{1,0}\left(\frac{iz}{2} \middle| \begin{matrix} - \\ (\frac{1}{2}, \frac{1}{2}), (0, \frac{1}{2}) \end{matrix} \right)$$

2. Hyperbolic Cosine Function

$$\cosh(z) = \frac{\sqrt{\pi}}{2} H_{0,2}^{1,0}\left(\frac{iz}{2} \middle| \begin{matrix} - \\ (0, \frac{1}{2}), (\frac{1}{2}, \frac{1}{2}) \end{matrix} \right)$$

3. Inverse Hyperbolic Sine Function

$$\sinh^{-1}(z) = \frac{1}{4\sqrt{\pi}} H_{2,2}^{1,2}\left(\frac{iz}{2} \middle| \begin{matrix} (1, \frac{1}{2}), (1, \frac{1}{2}) \\ (\frac{1}{2}, \frac{1}{2}), (0, \frac{1}{2}) \end{matrix} \right)$$

4. Inverse Hyperbolic Tangent Function

$$\tanh^{-1}(z) = \frac{-i}{4} H_{1,1}^{1,1}\left(iz \middle| \begin{matrix} (\frac{1}{2}, \frac{1}{2}) \\ (\frac{1}{2}, \frac{1}{2}) \end{matrix} \right)$$

4.1.4 Logarithmic Functions

$$\log(1 \pm z) = \pm H_{2,2}^{1,2}\left(\pm z \middle| \begin{matrix} (1, 1), (1, 1) \\ (1, 1), (0, 1) \end{matrix} \right)$$

4.1.5 Bessel Functions

- Bessel Function of the First Kind

$$J_\nu(z) = \frac{1}{2} H_{0,2}^{1,0}\left(\frac{z}{2} \middle| \begin{matrix} - \\ (\frac{\nu}{2}, \frac{1}{2}), (\frac{-\nu}{2}, \frac{1}{2}) \end{matrix} \right)$$

- Modified Bessel Function of the Second Kind

$$K_\nu(z) = \frac{1}{4} H_{0,2}^{2,0}\left(\frac{z}{2} \middle| \begin{matrix} - \\ (\frac{\nu}{2}, \frac{1}{2}), (\frac{-\nu}{2}, \frac{1}{2}) \end{matrix} \right)$$

4.1.6 Special Functions

- Riemann Zeta Function
- Jacobi Polynomials
- Legendre Polynomials
- Laguerre Polynomials
- Chebyshev Polynomials
- Hermite Polynomials
- Gegenbauer Polynomials
- Struve's Function
- Whittaker Functions
- Parabolic Cylinder Function
- Psi Function
- Error Function
- Complementary Error Function
- Mittag-Leffler Function
- Generalized Wright Function
- Meijer's G-Function

4.2 Properties of the H-Function

4.2.1 General Properties

- Reciprocal Argument

$$H_{p,q}^{m,n}\left(z \middle| \begin{matrix} (a_i, A_i) \\ (b_i, B_i) \end{matrix} \right) = H_{q,p}^{n,m}\left(\frac{1}{z} \middle| \begin{matrix} (a_i - b_i, B_i) \\ (1 - a_i, A_i) \end{matrix} \right)$$

- Argument to a real power

$$H_{p,q}^{m,n}\left(z^k \middle| \begin{matrix} (a_i, A_i) \\ (b_i, B_i) \end{matrix} \right) = \frac{1}{k} H_{p,q}^{m,n}\left(z \middle| \begin{matrix} (a_i, \frac{A_i}{k}) \\ (b_i, \frac{B_i}{k}) \end{matrix} \right)$$

- Multiplication by power of argument

$$z^k H_{p,q}^{m,n}\left(z \middle| \begin{matrix} (a_i, A_i) \\ (b_i, B_i) \end{matrix} \right) = H_{p,q}^{m,n}\left(z \middle| \begin{matrix} (a_i + kA_i, A_i) \\ (b_i + kB_i, B_i) \end{matrix} \right)$$

4.2.2 Derivatives of the H-Function

- Arbitrary r-th Derivative

$$\frac{d^r}{dz^r} H_{p,q}^{m,n}\left(z^k \middle| \begin{matrix} (a_i, A_i) \\ (b_i, B_i) \end{matrix} \right) = H_{p+1,q+1}^{m,n+1}\left(z^k \middle| \begin{matrix} (-r, k), (a_i - \frac{r}{k} A_i, A_i) \\ (b_i - \frac{r}{k} B_i, B_i), (r, k) \end{matrix} \right)$$

4.2.3 Integral Transforms of the H-Function

- Laplace Transform

$$\mathcal{L}\left\{H_{p,q}^{m,n}\left(cz\middle|\begin{array}{c}(a_i,A_i)\\(b_i,B_i)\end{array}\right)\right\}=\frac{1}{c}H_{q,p+1}^{n+1,m}\left(\frac{s}{c}\middle|\begin{array}{c}(1-b_i-B_i,B_i)\\(0,1),(1-a_i-A_i,A_i)\end{array}\right)$$

- Fourier Transform

$$\mathcal{F}\left\{H_{p,q}^{m,n}\left(cz\middle|\begin{array}{c}(a_i,A_i)\\(b_i,B_i)\end{array}\right)\right\}=\frac{1}{c}H_{q,p+1}^{n+1,m}\left(\frac{-is}{c}\middle|\begin{array}{c}(1-b_i-B_i,B_i)\\(0,1),(1-a_i-A_i,A_i)\end{array}\right)$$

5 Applications of the Fox H-Function

In this section, we discuss some of the applications of the Fox H-Function.

5.1 Probability Theory

Many distributions are special cases of the H-Distribution. Some of these are given in Fig. 5.1 [2].

The Moment Generating Function: Let $X \sim H$-Distributed Random Variable (RV). Then the Moment Generating Function of X is also a H-Function.

Cumulative Distribution Function: The PDF, CDF and MGF of H-Distributed RVs are all H-Functions.

Sum of Random Variables: Let $X_i \sim H$-Distributed Independent Random Variables $\forall i$. Then, the MGF of $Y := \sum_i X_i$ is also H-Distributed. The RV itself can be written in terms of the multivariate H-Function.

Product of Random Variables: Let $X_i \sim H$-distributed Independent Random Variables $\forall i$. Then,

$Z := \prod_i X_i$ is also H-Distributed.

5.2 Applications in Physical Sciences

Other areas in which H-Function is used include:

- **Wireless Communications:** The H-Distribution is used to model fading in wireless channels.
- **Statistical and Stochastic Modelling:** Based on the H-Distribution and its properties.
- **Integral Transform Theory:** Many Integral Transforms (Laplace, Fourier etc.) are special cases of the H-Transform.
- **Fractional Calculus:** Some fractional derivatives of the H-Function are also H-Functions.
- **Integro-Differential Equations:** Based on the fact that integrals and derivatives of H-Functions are H-Functions.
- **Astrophysics:** Analytic Solar and Stellar Models, Gravitational Instability Problem, Nonextensive Statistical Mechanics.

One-Sided Gaussian [2, Sec. 2.2.1.4]

$$p_{\gamma_\ell}(\gamma) = \sqrt{\frac{2}{\pi\bar{\gamma}_\ell}} \exp\left(-\frac{\gamma^2}{2\bar{\gamma}_\ell}\right), = \frac{1}{2\sqrt{\pi\bar{\gamma}_\ell}} \mathrm{H}^{1,0}_{0,1}\left[\frac{\gamma}{2\bar{\gamma}_\ell} \,\middle|\, \begin{matrix} \overline{\quad} \\ (-\frac{1}{2}, 1) \end{matrix}\right],$$

where $\bar{\gamma}_\ell$ is the average power (i.e., $\bar{\gamma}_\ell \geq 0$).

Exponential [2, Eq. (2.7)]

$$p_{\gamma_\ell}(\gamma) = \frac{1}{\bar{\gamma}_\ell} \exp\left(-\frac{\gamma}{\bar{\gamma}_\ell}\right) = \frac{1}{\bar{\gamma}_\ell} \mathrm{H}^{1,0}_{0,1}\left[\frac{\gamma}{\bar{\gamma}_\ell} \,\middle|\, \begin{matrix} \overline{\quad} \\ (0, 1) \end{matrix}\right],$$

where $\bar{\gamma}_\ell$ is the average power (i.e., $\bar{\gamma}_\ell \geq 0$).

Gamma [2, Eq. (2.21)]

$$p_{\gamma_\ell}(\gamma) = \frac{1}{\Gamma(m_\ell)} \left(\frac{m_\ell}{\bar{\gamma}_\ell}\right)^{m_\ell} \gamma^{m_\ell - 1} \exp\left(-\frac{m_\ell \gamma}{\bar{\gamma}_\ell}\right), = \frac{m_\ell}{\Gamma(m_\ell)\bar{\gamma}_\ell} \mathrm{H}^{1,0}_{0,1}\left[\frac{m_\ell}{\bar{\gamma}_\ell}\gamma \,\middle|\, \begin{matrix} \overline{\quad} \\ (m_\ell - 1, 1) \end{matrix}\right],$$

where $\bar{\gamma}_\ell$ is the average power, and where m_ℓ ($0.5 \leq m_\ell$) denotes the fading figure. Moreover, $\Gamma(\cdot)$ is the Gamma function [8, Sec. 8.31].

Weibull [2, Eq. (2.27)]

$$p_{\gamma_\ell}(r) = \xi_\ell \left(\frac{\omega_\ell}{\bar{\gamma}_\ell}\right)^{\xi_\ell} r^{\xi_\ell - 1} \exp\left(-\left(\frac{\omega_\ell}{\bar{\gamma}_\ell}\right)^{\xi_\ell} r^{\xi_\ell}\right), = \frac{\omega_\ell}{\bar{\gamma}_\ell} \mathrm{H}^{1,0}_{0,1}\left[\frac{\omega_\ell}{\bar{\gamma}_\ell}\gamma \,\middle|\, \begin{matrix} \overline{\quad} \\ (1 - 1/\xi_\ell, 1/\xi_\ell) \end{matrix}\right],$$

where $\omega_\ell = \Gamma(1 + 1/\xi_\ell)$ and where ξ_ℓ ($0 < \xi_\ell$) denotes the fading shaping factor. Moreover, $\bar{\gamma}_\ell$ is the average power.

Generalized Gamma [38]

$$p_{\gamma_\ell}(\gamma) = \frac{\xi_\ell \left(\frac{\beta_\ell}{\bar{\gamma}_\ell}\right)^{m_\ell \xi_\ell} \gamma^{m_\ell \xi_\ell - 1}}{\Gamma(m_\ell)} \exp\left(-\left(\frac{\beta_\ell}{\bar{\gamma}_\ell}\gamma\right)^{\xi_\ell}\right), = \frac{\beta_\ell}{\Gamma(m_\ell)\bar{\gamma}_\ell} \mathrm{H}^{1,0}_{0,1}\left[\frac{\beta_\ell}{\bar{\gamma}_\ell}\gamma \,\middle|\, \begin{matrix} \overline{\quad} \\ (m_\ell - \frac{1}{\xi_\ell}, \frac{1}{\xi_\ell}) \end{matrix}\right],$$

Extended Generalized Gamma [27]

$$p_{\gamma_\ell}(\gamma) = \frac{\xi_\ell \left(\frac{\beta_\ell \beta_{s\ell}}{\bar{\gamma}_{s\ell}}\right)^{\xi_\ell m_\ell}}{\Gamma(m_\ell)\Gamma(m_{s\ell})} \gamma^{\xi_\ell m_\ell - 1} \times \Gamma\left(m_{s\ell} - m_\ell \frac{\xi_\ell}{\xi_{s\ell}}, 0, \left(\frac{\beta_\ell \beta_{s\ell}}{\bar{\gamma}_{s\ell}}\right)^{\xi_\ell} \gamma^{\xi_\ell}, \frac{\xi_\ell}{\xi_{s\ell}}\right), = \frac{\mathrm{H}^{2,0}_{0,2}\left[\frac{\beta_\ell \beta_{s\ell}}{\bar{\gamma}_{s\ell}}\gamma \,\middle|\, \begin{matrix} \overline{\quad} \\ (m_\ell - 1/\xi_\ell, 1/\xi_\ell), (m_{s\ell} - 1/\xi_{s\ell}, 1/\xi_{s\ell}) \end{matrix}\right]}{\frac{\Gamma(m_\ell)\Gamma(m_{s\ell})\bar{\gamma}_{s\ell}}{\beta_\ell \beta_{s\ell}}},$$

Fox's H distribution [30, Eq. (3.1)], [22]

$$p_{\gamma_\ell}(\gamma) = \mathcal{K}_\ell \mathrm{H}^{m,n}_{p,q}\left[\mathcal{G}_\ell \gamma \,\middle|\, \begin{matrix} (a_i, \alpha_i)_{i=1,2,\ldots,p} \\ (b_j, \beta_j)_{j=1,2,\ldots,q} \end{matrix}\right],$$

where $\mathcal{K}_\ell \in \mathbb{R}$ and $\mathcal{G}_\ell \in \mathbb{R}$ are such two numbers that $\int_0^\infty p_{\gamma_\ell}(\gamma)\, d\gamma = 1$.

Figure 2: Distributions of H-Random Variables

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[3] Dennis G. Zill and Patrick Shanahan, *Complex analysis*, 3rd ed., Jones and Bartlett Publishers, Inc., USA, 2013.